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Partition function for multi-cut matrix models

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Abstract
We consider the partition function Z_N of a random matrix model with polynomial potential $V(\xi) = t_1\xi + t_2\xi^2 + \cdots + t_{2d}\xi^{2d}$. It is known that the second logarithmic derivative of Z_N with respect to the times t_k can be expressed in terms of the recurrence coefficients of the related orthogonal polynomials. An explicit formula for the recurrence coefficients of the orthogonal polynomials in the limit $N \to \infty$ for multi-cut regular $V(\xi)$ has been derived in [10] through the Riemann–Hilbert approach. The expression for Z_N in the limit $N \to \infty$ has been derived in [7] through a mean-field approach. We show that the above asymptotic formulae satisfy the same relations that hold for finite N.

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1. Introduction

We consider the partition function of a random matrix model,

$$Z_N = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \prod_{1\leqslant j<k\leqslant N} (\xi_j - \xi_k)^2 \exp\left(-N\sum_{j=1}^{N} V(\xi_j)\right) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_N = N! \prod_{n=0}^{N-1} h_n, \tag{1.1}$$

where $V(\xi)$ is a polynomial,

$$V(\xi) = \sum_{j=1}^{2d} t_j \xi^j, \qquad t_{2d} > 0, \tag{1.2}$$

and h_n are the normalization constants of the orthogonal polynomials $\pi_n(\xi)$ on the line with respect to the weight $\mathrm{e}^{-NV(\xi)}$,

$$\int_{-\infty}^{\infty} \pi_n(\xi)\pi_m(\xi)\,\mathrm{e}^{-NV(\xi)}\,\mathrm{d}\xi = h_n\delta_{nm}, \qquad \pi_n(\xi) = \xi^n + \cdots. \tag{1.3}$$

In this work we are interested in the asymptotic expansion of the partition function as $N \to \infty$ in the so-called multi-gap case, namely when the support of the equilibrium measure $\psi(\xi)\,\mathrm{d}\xi$, which solves the variational problem,

$$F_0 = \operatorname*{Min}_{\{\psi \geqslant 0, \int \psi \mathrm{d}\xi = 1\}} \left[-\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \log|\xi - \eta|\psi(\xi)\psi(\eta)\,\mathrm{d}\xi\,\mathrm{d}\eta + \int_{-\infty}^{+\infty} V(\xi)\psi(\xi)\,\mathrm{d}\xi \right], \tag{1.4}$$

consists of many intervals. In the one-cut regular case

$$-\frac{1}{N^2}\log Z_N \propto F_0 + \frac{1}{N^2}F_1 + \frac{1}{N^4}F_2 + \cdots,$$

that is, the logarithmic of the partition function has a regular asymptotic expansion in powers of $1/N^2$ [3, 5, 14] and the terms $F_1, F_2, \ldots$ can be determined from F_0 [16].

Define the orthonormal polynomials as $p_n(\xi) = \frac{1}{\sqrt{h_n}}\pi_n(\xi)$; then

$$\int_{-\infty}^{\infty} p_n(\xi)p_m(\xi)\,\mathrm{e}^{-NV(\xi)}\,\mathrm{d}\xi = \delta_{nm}. \tag{1.5}$$

The polynomials $p_n(\xi)$ satisfy the three-term recurrence relation

$$zp_n(\xi) = \gamma_{n+1}p_{n+1}(\xi) + \beta_n p_n(\xi) + \gamma_n p_{n-1}(\xi), \qquad \gamma_n = \sqrt{\frac{h_n}{h_{n-1}}}. \tag{1.6}$$

The recurrence coefficients evolve with respect to the times t_k according to the equations [1, 12, 15, 18]

$$\frac{1}{N}\frac{\partial \ln h_n}{\partial t_k} = -[Q^k]_{nn}, \tag{1.7}$$

$$\frac{1}{N}\frac{\partial \gamma_n}{\partial t_k} = \frac{\gamma_n}{2}([Q^k]_{n-1,n-1} - [Q^k]_{nn}), \tag{1.8}$$

$$\frac{1}{N}\frac{\partial \beta_n}{\partial t_k} = \gamma_n[Q^k]_{n,n-1} - \gamma_{n+1}[Q^k]_{n+1,n}, \tag{1.9}$$

where $[Q^k]_{nm}$ denotes the nm th element of the matrix Q^k, and Q takes the form

$$Q = \begin{pmatrix} \beta_0 & \gamma_1 & 0 & 0 & 0 & \cdots \\ \gamma_1 & \beta_1 & \gamma_2 & 0 & 0 & \cdots \\ 0 & \gamma_2 & \beta_2 & \gamma_3 & 0 & \cdots \\ 0 & 0 & \gamma_3 & \beta_3 & \gamma_4 & \cdots \\ 0 & 0 & 0 & \gamma_4 & \beta_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{1.10}$$

The following relations which connect the derivatives with respect to t_k's of the partition functions Z_N and the coefficients γ_N, β_N and h_N were derived in [5]:

$$\frac{1}{N^2}\frac{\partial^2 \ln Z_N}{\partial t_1^2} = \gamma_N^2, \qquad \frac{1}{N^2}\frac{\partial^2 \ln Z_N}{\partial t_1 \partial t_2} = \gamma_N^2(\beta_{N-1} + \beta_N), \tag{1.11}$$

$$\frac{1}{N^2}\frac{\partial^2 \ln Z_N}{\partial t_2^2} = \gamma_N^2\left(\gamma_{N-1}^2 + \gamma_{N+1}^2 + \beta_N^2 + 2\beta_N\beta_{N-1} + \beta_{N-1}^2\right). \tag{1.12}$$

Similar formulae can be obtained for the derivatives with respect to higher times t_k, $k \geqslant 2$. The above formulae have been derived by Bleher and Its [5] for obtaining the full asymptotic

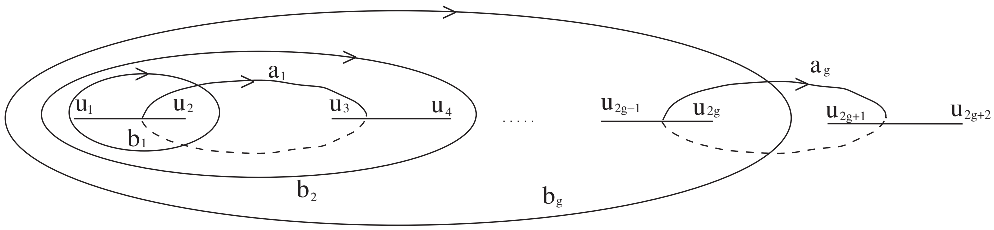

Figure 1. The homology basis.

expansion of the partition function Z_N in powers of $1/N^2$ in the one-cut regular case. The same result has been proved earlier [14], using a different approach, to make the Bessis–Itzykson–Zuber topological expansion rigorous [3].

The behaviour of the large N limit of the coefficients γ_N and β_{N-1} when the support of the equilibrium measure consists of many intervals has been obtained by Deift *et al* [10] through a Riemann–Hilbert approach. The asymptotic formulae can be described in the following way. Let us assume that the support J of the equilibrium measure $\psi(\xi)\,\mathrm{d}\xi$ consists of $g+1$ intervals, namely $J=\cup_{k=0}^{g}(u_{2k+1},u_{2k+2})$, $g\leqslant d-1$, and introduce the Riemann surface X of genus g associated with the curve $y^2=R(\xi)$ where $R(\xi)=\prod_{k=1}^{2g+2}(\xi-u_k)$. X is considered as a double-sheeted covering of the complex plane. Introduce a basis of canonical cycles $\{a_1,\ldots,a_g,b_1,\ldots,b_g\}$, as shown in figure 1, and the corresponding basis $\boldsymbol{\omega}=(\omega_1,\ldots,\omega_g)$ of normalized holomorphic differentials

$$\int_{a_j}\omega_i=\delta_{ij},\qquad i,j=1\ldots,g.$$

The corresponding period matrix B takes the form $B_{ij}=\int_{b_j}\omega_i, i,j=1,\ldots,g$, and the θ-function is defined as $\theta(\boldsymbol{z})=\sum_{\boldsymbol{n}\in\mathbb{C}^g}\exp(\pi\mathrm{i}\langle\boldsymbol{n},B\boldsymbol{n}\rangle+2\pi\mathrm{i}\langle\boldsymbol{z},\boldsymbol{n}\rangle)$. Next, we introduce the following vectors $\boldsymbol{\Omega}=(\Omega_1,\Omega_2,\ldots,\Omega_g)$:

$$\Omega_j=2\pi\int_{u_{2j+1}}^{u_{2g+2}}\psi(\xi)\,\mathrm{d}\xi \tag{1.13}$$

and

$$\boldsymbol{v}_+=\int_{u_{2g+2}}^{\infty^1}\boldsymbol{\omega},\qquad \boldsymbol{v}=\int_{\infty^2}^{\infty^1}\boldsymbol{\omega}=2\boldsymbol{v}_+,$$

where $\infty^{1,2}$ are the points at infinity on the first and second sheets of X, respectively. The first sheet corresponds to the positive sign of $\sqrt{R(\xi)}$ as $\xi\to\infty$. With the above notation, the behaviour of γ_N and β_{N-1} as $N\to\infty$ is given by [10]

$$\gamma_N^2=\left(\gamma_N^0\right)^2+O(1/N),\qquad \beta_{N-1}=\beta_{N-1}^0+O(1/N),$$

where[1]

$$\left(\gamma_N^0\right)^2=\left(\frac{1}{4}\sum_{j=1}^{g+1}(u_{2j}-u_{2j-1})\right)^2\frac{\theta(\mathbf{0})^2}{\theta\left(\frac{N}{2\pi}\boldsymbol{\Omega}\right)^2}\frac{\theta\left(\boldsymbol{v}+\frac{N}{2\pi}\boldsymbol{\Omega}\right)\theta\left(\boldsymbol{v}-\frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta(\boldsymbol{v})^2}, \tag{1.14}$$

[1] We remark that formulae (1.14) and (1.15) look slightly different from those derived in [10] because we apply the identity

$$\boldsymbol{d}+\boldsymbol{v}_+=0,$$

where $\boldsymbol{d}$ is the vector introduced in (1.30) of [10].

$$\beta_{N-1}^0 = \sum_{k=1}^{g} \partial_{z_k} \log \frac{\theta\left(\boldsymbol{v} + \frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta\left(\frac{N}{2\pi}\boldsymbol{\Omega}\right)\theta(\boldsymbol{v})} \omega_k(\infty^1), \tag{1.15}$$

with z_k the kth component of the argument of the θ-function, and $\omega_k(\infty^1) := \frac{\omega_k(\xi)}{\mathrm{d}s}\big|_{s=0}$, $s = 1/\xi$. Equivalent formulae have been obtained in [7] by a mean-field approach. We remark that the error term $O(1/N)$ in (1.14) and (1.15) holds only in the regular case, but not in the singular cases where the equilibrium measure vanishes at the interior point of J or to higher order at the endpoints of J.

In this paper we show, at the algebraic level, that the asymptotic formulae (1.14) and (1.15) satisfy

$$\frac{1}{N^2}\frac{\partial^2}{\partial t_1^2} \log\left[\mathrm{e}^{-N^2 F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right] = \gamma_N^0 + O(1/N), \tag{1.16}$$

$$\frac{1}{N^2}\frac{\partial^2}{\partial t_1 \partial t_2} \log\left[\mathrm{e}^{-N^2 F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right] = \left(\gamma_N^0\right)^2\left(\beta_{N-1}^0 + \beta_N^0\right) + O(1/N), \tag{1.17}$$

$$\frac{1}{N^2}\frac{\partial^2}{\partial t_2^2} \log\left[\mathrm{e}^{-N^2 F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right] = \left(\gamma_N^0\right)^2\Big(\left(\gamma_{N-1}^0\right)^2 + \left(\gamma_{N+1}^0\right)^2 + \left(\beta_N^0\right)^2 + 2\beta_N^0\beta_{N-1}^0 + \left(\beta_{N-1}^0\right)^2\Big) + O(1/N), \tag{1.18}$$

where F_0 and $\boldsymbol{\Omega}$ have been defined in (1.4) and (1.13), respectively. Similar relations hold for the derivatives with respect to the other parameters t_k, $2 \leqslant k \leqslant 2d$. The above identities are in agreement with the derivation of the asymptotic behaviour of the partition function in the large N limit obtained in [7] using a saddle point argument:

$$Z_N \simeq \mathrm{e}^{-N^2 F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right).$$

A similar formula has been obtained in the context of the zero dispersion limit of the Korteweg de Vries equation [24].

We remark that the derivatives of the free energy F_0 with respect to the times t_k have been obtained in many papers and also in two-matrix models introducing the concept of filling fractions, that is, fixing the density of the eigenvalues n_k in each interval (u_{2k-1}, u_{2k}), $k = 1, \ldots, g+1$, namely by adding to the variational problem (1.4) the term $\sum_{k=1}^{g+1} \lambda_k \int_{u_{2k-1}}^{u_{2k}} (\psi(\xi)\,\mathrm{d}\xi - n_k)$ where λ_k are the Lagrange multipliers (see, e.g., [8, 2]). In this paper, we follow a different approach evaluating the derivatives with respect to the times t_k directly on F_0.

Despite the relations (1.14) and (1.15) being derived for a fixed external field $V(\xi)$, we assume that such a formula holds true while varying $V(\xi)$ in a sufficiently small range. This is to stress that relations (1.16)–(1.18) are formal identities and represent a first step towards the rigorous mathematical derivation in the spirit of [5], of the expression of the partition function Z_N in the large N limit, when the support of the eigenvalues is distributed on many intervals. The same result could possibly be obtained exploiting the relation between the partition function Z_N and the isomonodromic τ-function [1].

This paper is organized as follows. In the first section we present the necessary ingredients to compute the derivatives on the lhs of (1.16)–(1.18) and in the second section we reduce such derivatives to the terms on the rhs of (1.16)–(1.18).

2. Times derivative of the equilibrium measure and F_0

In this section we compute the derivatives with respect to t_k's of the equilibrium measure $\psi(\xi)\,\mathrm{d}\xi$ and of the planar limit F_0 of the free energy.

The minimization problem (1.4) has been widely studied and it is reduced to the following Euler–Lagrange equations:

$$L\psi(\xi) - V(\xi) = l \qquad \text{where} \quad \psi > 0, \tag{2.1}$$

$$L\psi(\xi) - V(\xi) \leqslant l \qquad \text{where} \quad \psi = 0, \tag{2.2}$$

where l is the Lagrange multiplier and

$$L\psi(\xi) = \int_{-\infty}^{+\infty} \log|\xi - \mu|\psi(\mu)\,\mathrm{d}\mu. \tag{2.3}$$

It can be shown [9] that ψ is the minimizer iff ψ is a non-negative function that satisfies the variational conditions (2.1)–(2.2) and the constraint

$$\int_{-\infty}^{+\infty} \psi(\xi)\,\mathrm{d}\xi = 1. \tag{2.4}$$

Assuming that the support J of the equilibrium measure consists of $g+1$ intervals, namely $J = \cup_{k=0}^{g}(u_{2k+1}, u_{2k+2})$, the equilibrium measure takes the form

$$\psi(\xi) = \frac{1}{\pi\mathrm{i}}\Phi(\xi)\sqrt{R(\xi)}_+ \tag{2.5}$$

$$R(\xi) = \prod_{k=1}^{2g+2}(\xi - u_k) \tag{2.6}$$

$$\Phi(\xi) = -\frac{1}{2\pi\mathrm{i}}\oint \frac{V'(s)}{\sqrt{R(s)}}\frac{\mathrm{d}s}{s-\xi}, \tag{2.7}$$

where $\sqrt{R(\xi)}_+$ denotes the boundary value on J from the above and $\sqrt{R(\xi)}$ behaves like ξ^{g+1} as $\xi \to \infty$. The contour integral in (2.7) is a closed clockwise loop around $J \cup \xi$. The end points of the support are determined from (2.4), the moment conditions

$$\frac{1}{2\pi\mathrm{i}}\oint \frac{V'(s)s^k}{\sqrt{R(s)}}\,\mathrm{d}s = 0, \qquad k = 0, \ldots, g, \tag{2.8}$$

and the conditions

$$\int_{u_{2k+1}}^{u_{2k}} \mathrm{d}\xi\,\Phi(\xi)\sqrt{R(\xi)} = 0, \qquad k = 1, \ldots, g. \tag{2.9}$$

The equilibrium measure is called regular (otherwise singular) [10] if $\Phi(\xi) \neq 0$ for $\xi \in \bar{J}$, where $\bar{J}$ is the closure of J, and the inequality (2.2) is strict for $\xi \in \mathbb{R}\backslash J$. In the following, we assume that the equilibrium measure is regular and we call the corresponding external field $V(\xi)$ regular. The variation of the equilibrium measure with respect to the end points of the support has been obtained in [23] where it is shown that the regular behaviour of the equilibrium measure is generic for a real analytic external field. In the following, we are interested in obtaining the derivatives with respect to the times t_k of the equilibrium measure $\psi(\xi)$ and we suppose that the variation of t_k is sufficiently small so that the equilibrium measure remains regular. For the purpose, we rewrite equations (2.8) and (2.9) as the zeros of a meromorphic 1-form on the Riemann surface X. This approach introduced by Krichever [22] is well known in the theory of Whitham equations. On the Riemann surface X of genus g, we introduce the normalized meromorphic 1-forms $\sigma_k(\xi)$ with a pole at the point $\infty^1 = (\infty, +\infty)$ of order $k+1$, $k \geqslant 1$, and the differential $\sigma_0(\xi)$ with first-order poles at the points $\infty^{1,2} = (\infty, \pm\infty)$ with residue ± 1 respectively, namely

$$\sigma_k(\xi) = \frac{1}{2}\xi^{k-1}\,\mathrm{d}\xi + \frac{1}{2}\frac{P_k(\xi)}{\sqrt{R(\xi)}}\,\mathrm{d}\xi, \qquad k \geqslant 1$$

$$\sigma_0(\xi) = \frac{P_0(\xi)}{\sqrt{R(\xi)}}\,\mathrm{d}\xi,$$

where $P_k(\xi) = \xi^{g+k} + a_{g,1}\xi^{g+k-1} + \cdots + a_{g,g+k}$ and its coefficients are uniquely determined by

$$\frac{P_k(\xi)}{\sqrt{R(\xi)}} = \xi^{k-1} + O\left(\frac{1}{\xi^2}\right) \qquad \text{for large } |\xi|, \tag{2.10}$$

and

$$\int_{a_j} \sigma_k(\xi) = 0, \qquad j = 1, 2, \ldots, g. \tag{2.11}$$

Let us introduce the differential

$$\Omega(\xi) = \sum_{k=1}^{d} k t_k \sigma_k(\xi) - \sigma_0(\xi). \tag{2.12}$$

The following identity holds.

Theorem 2.1. *Equations (2.8) and (2.9) are equivalent to the equations*

$$\left.\frac{\Omega(\xi)}{\mathrm{d}\eta}\right|_{\eta=0} = 0, \qquad \eta = \sqrt{\xi - u_k}, \qquad k = 1\ldots, 2g+2, \tag{2.13}$$

where the differential $\Omega(\xi)$ has been defined in (2.12).

Proof. The proof is similar to that of the KdV case [19, 20]. The differential Ω can be written in the form

$$\Omega(\xi) = \frac{1}{2}V'(\xi) + \sqrt{R(\xi)}\Phi(\xi)\,\mathrm{d}\xi + \frac{Q(\xi)}{\sqrt{R(\xi)}}\,\mathrm{d}\xi, \tag{2.14}$$

where $\Phi(\xi)$ has been defined in (2.7) and

$$Q(\xi) = \sum_{i=1}^{2g+2}\left[\prod_{l=1, l\neq i}^{2g+2}(\xi - u_l)\right]\partial_{u_i} q_g(\boldsymbol{u}) - P_0(\xi) + \sum_{k=1}^{g} P_k(\xi) k \sum_{l=0}^{g-k}\Gamma_l(\boldsymbol{u}) q_{k+l}(\boldsymbol{u}), \tag{2.15}$$

where $\Gamma_l(\boldsymbol{u})$'s come from the expansion

$$\sqrt{R(\mu)} = \mu^{g+1}\left[\Gamma_0(\boldsymbol{u}) + \frac{\Gamma_1(\boldsymbol{u})}{\mu} + \frac{\Gamma_2(\boldsymbol{u})}{\mu^2} + \cdots\right]. \tag{2.16}$$

The function q_k is

$$q_k(\boldsymbol{u}) = \frac{\mathrm{i}}{2\pi}\int_J \frac{V(\mu)\mu^{g-k}}{R(\mu)}\,\mathrm{d}\mu, \qquad k = 1, \ldots, g. \tag{2.17}$$

Equations (2.13) and (2.14) imply that the polynomial $Q(\xi)$ is identically zero, namely

$$Q(\xi) \equiv 0.$$

Indeed, the first two terms in (2.14) are automatically zero at the points u_i, $i = 1, \ldots, 2g+2$. So it follows that the polynomial $Q(\xi)$ of degree $2g+1$ must have $2g+2$ zeros. Therefore, it is identically zero. Putting equal to zero the coefficients of $Q(\xi)$ from degree $2g+1$ to degree

$g + 1$ is equivalent to equation (2.8). To show that equations (2.13) imply (2.9), it is sufficient to observe that on the solution of (2.13) the conditions

$$\int_{a_i} \Omega(\xi) = 0, \qquad i = 1, \ldots, g,$$

take the form (2.9). □

On the solution of (2.13), because of (2.14), the equilibrium measure can be written in the form

$$\psi(\xi)\,\mathrm{d}\xi = \mathrm{Re}\left(\frac{\Omega(\xi)}{\pi \mathrm{i}}\right), \tag{2.18}$$

where Re is the real part of the differential $\Omega(\xi)$. Here and in the rest of the paper, we assume that $\sqrt{R(\xi)}$ appearing in the Abelian differentials coincides with $\sqrt{R(\xi)}_+$ for $\xi \in \mathbb{R}$. From (2.18), it is clear that

$$\int_J \psi(\xi)\,\mathrm{d}\xi = \int_J \mathrm{Re}\left(\frac{\Omega(\xi)}{\pi \mathrm{i}}\right) = 1,$$

because the differentials $\sigma_k, k \geqslant 1$, have zero residue at ∞^+, and σ_0 has residue equal to 1. The above formulation enables one to evaluate the derivatives with respect to the times t_k of the equilibrium measure and of the integrals Ω_j in a straightforward way using the approach of Krichever [22].

Proposition 2.2. *The following relations hold:*

$$\frac{\partial}{\partial t_k}\psi(\xi)\,\mathrm{d}\xi = k\,\mathrm{Re}\left(\frac{\sigma_k(\xi)}{\pi \mathrm{i}}\right), \qquad k \geqslant 1, \tag{2.19}$$

$$\frac{\partial}{\partial t_k}\Omega_j = -2\pi \operatorname*{Res}_{\xi=\infty}(\xi^k \omega_j(\xi)), \qquad j = 1, \ldots, g. \tag{2.20}$$

Proof. We observe that

$$\frac{\partial}{\partial t_k}\Omega(\xi) = k\sigma_k(\xi) + \left[\sum_{j=1}^{d} j t_j \frac{\partial}{\partial t_k}\sigma_j(\xi) - \frac{\partial}{\partial t_k}\sigma_0(\xi)\right].$$

The expression in brackets is a normalized Abelian differential which does not have a pole at infinity because the principal part of the differentials $\sigma_j, j \geqslant 0$, is independent of t_k's, and it does not have a pole at u_j, $j = 1, \ldots, 2g + 2$, in view of (2.13). Hence, the differential in the bracket is a holomorphic differential with all the a-periods equal to zero and therefore it is identically zero, so

$$\frac{\partial}{\partial t_k}\Omega(\xi) = k\sigma_k(\xi). \tag{2.21}$$

Relation (2.19) follows from the above relation and (2.18). To prove (2.20), it is sufficient to observe that, by (2.18)

$$\Omega_j = 2\pi \int_{u_{2j+1}}^{u_{2g+2}} \psi(\xi)\,\mathrm{d}\xi = 2\pi - \mathrm{Im} \int_{b_j} \Omega(\xi),$$

where Im is the imaginary part, by (2.21) and by the Riemann bilinear relations

$$\frac{\partial}{\partial t_k}\Omega_j = -k\,\mathrm{Im} \int_{b_j} \sigma_k(\xi) = 2\pi \operatorname*{Res}_{\xi=\infty}(\xi^k \omega_j(\xi)).$$

□

From (2.19) the derivative of F_0 with respect to the times t_k can be evaluated in a straightforward way.

Proposition 2.3. *The following relations hold:*

$$\frac{\partial^2 F_0}{\partial t_k \partial t_j} = \frac{j}{\pi \mathrm{i}} \int_J \xi^k \sigma_j(\xi) = \frac{k}{\pi \mathrm{i}} \int_J \xi^j \sigma_k(\xi), \qquad k, j \geqslant 1. \tag{2.22}$$

Proof. From (2.19), we obtain

$$\partial_{t_k} F_0 = -\frac{2k}{\pi \mathrm{i}} \int_J \int_J \log|\xi - \eta| \sigma_k(\xi) \psi(\eta)\, \mathrm{d}\eta + \frac{k}{\pi \mathrm{i}} \int_J V(\xi) \sigma_k(\xi) + \int_J \xi^k \psi(\xi)\, \mathrm{d}\xi$$
$$= \frac{lk}{\pi \mathrm{i}} \int_J \sigma_k(\xi) + \int_J \xi^k \psi(\xi)\, \mathrm{d}\xi,$$

where l is the Lagrange multiplier. Next, performing the t_j derivative and observing that $\int_J \sigma_k(\xi) = 0$ and $\partial_{t_j} \int_J \sigma_k(\xi) = 0$, we obtain

$$\frac{\partial^2 F_0}{\partial t_k \partial t_j} = \frac{j}{\pi \mathrm{i}} \int_J \xi^k \sigma_j(\xi). \tag{2.23}$$

The identity $j \int_J \xi^k \sigma_j(\xi) = k \int_J \xi^j \sigma_k(\xi)$ follows from the symmetry of F_0 with respect to the times derivatives. □

In the following, we aim at writing relation (2.22) in a clear symmetric form using the so-called canonical symmetric 2-differential $\mathcal{B}(P, Q)$, $P, Q, \in X$. $\mathcal{B}(P, Q)$ is the canonical symmetric 2-form which is uniquely determined by the following conditions:

- $\mathcal{B}(P, Q)$ is symmetric in its arguments;
- all the a-periods of $\mathcal{B}(P, Q)$ with respect to any of its two variables vanish. The period with respect to the variable P or Q, along the b_k cycle, is equal to $2\pi \mathrm{i} \omega_k(Q)$ or $2\pi \mathrm{i} \omega_k(P)$, respectively;
- $\mathcal{B}(P, Q)$ has a double pole along the diagonal with the following local behaviour:

$$\mathcal{B}(P, Q) = \left(\frac{1}{(x(P) - x(Q))^2} + O(1) \right) \mathrm{d}x(P)\, \mathrm{d}x(Q), \tag{2.24}$$

  where x is a local coordinate.

The Abelian differential σ_k satisfies the relation

$$\sigma_k(Q) = -\frac{1}{k} \operatorname*{Res}_{P=\infty^1} (\eta^k \mathcal{B}(P, Q)), \qquad P = (\eta, w) \in X, \tag{2.25}$$

where $\infty^1 = (\infty, +\infty)$, $Q = (\xi, y) \in X$. So the identity on the rhs of (2.22) corresponds to

$$\int_J \xi^j \operatorname*{Res}_{P=\infty^1} (\eta^k \mathcal{B}(P, Q)) = \int_J \xi^k \operatorname*{Res}_{P=\infty^1} (\eta^j \mathcal{B}(P, Q)).$$

Therefore, by (2.25), relation (2.22) can be written in the form

$$\frac{\partial^2 F_0}{\partial t_k \partial t_j} = - \operatorname*{Res}_{Q=\infty^1} \operatorname*{Res}_{P=\infty^1} (\eta^k \xi^j \mathcal{B}(P, Q)), \qquad P = (\eta, w) \in X, \quad Q = (\xi, y) \in X. \tag{2.26}$$

We would like to stress that relation (2.22) is well known in the theory of Hermitian one-matrix models with a filling fraction. The derivation of the same formulae in this context has to follow a different approach.

Combining propositions (2.2) and (2.26), we arrive at the following result.

Theorem 2.4. *The following relations are satisfied:*

$$\frac{1}{N^2}\frac{\partial^2}{\partial t_k \partial t_j}\log\left[\mathrm{e}^{-N^2F_0}\theta\left(\frac{N\Omega}{2\pi}\right)\right] = \operatorname*{Res}_{P=\infty^1}\operatorname*{Res}_{Q=\infty^1}\left(\xi^j\eta^k\left[\mathcal{B}(P,Q)+\sum_{n,m=1}^{g}\frac{\partial^2}{\partial z_n\partial z_m}\log\theta\right.\right.$$
$$\left.\left.\times\left(\frac{N\Omega}{2\pi}\right)\omega_m(P)\omega_n(Q)\right]\right)+O\left(\frac{1}{N}\right), \tag{2.27}$$

where $P=(\xi,y)\in X$ *and* $Q=(\eta,w)\in X$.

The above theorem is the first step towards the proof of identities (1.16)–(1.18).

3. Formal identities for the asymptotic of the recurrence coefficients

Let us recall the basic steps of the Riemann–Hilbert approach to the asymptotic analysis of the orthogonal polynomial following the scheme of [10]. The principal observation [18] is that the orthogonal polynomials $P_n(\xi)$ admit the representation

$$P_n(\xi) = Y_{11}(\xi,n), \tag{3.1}$$

where the 2×2 matrix function $Y(\xi,n)$ is the (unique) solution of the following Riemann–Hilbert problem (RHP).

(1) $Y(\xi,n)$ is analytic for $\xi\in\mathbb{C}\backslash\mathbb{R}$, and it has continuous limits, $Y_+(\xi,n)$ and $Y_-(\xi,n)$, from above and below the real line, respectively,

$$Y_{n\pm}(\xi) = \lim_{\xi'\to\xi,\pm\mathrm{Im}\xi'>0} Y(\xi',n).$$

(2) $Y(\xi,n)$ satisfies the jump condition on the real line,

$$Y_+(\xi,n) = Y_-(\xi,n)\begin{pmatrix}1 & \mathrm{e}^{-NV(\xi)}\\ 0 & 1.\end{pmatrix}. \tag{3.2}$$

(3) As $\xi\to\infty$, the function $Y(\xi,n)$ has the following uniform asymptotic expansion:

$$Y(\xi,n)\sim\left(I+\sum_{k=1}^{\infty}\frac{Y_k(n)}{\xi^k}\right)\xi^{n\sigma_3},\qquad \xi\to\infty, \tag{3.3}$$

where

$$\sigma_3 = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In addition to equation (3.1), the recurrence coefficients γ_n and β_{n-1} can also be evaluated directly via $Y(\xi,n)$ by the formulae

$$\gamma_n^2 = (Y_1(n))_{21}(Y_1(n))_{12}, \tag{3.4}$$

$$\beta_{n-1} = \frac{(Y_2(n))_{21}}{(Y_1(n))_{21}} - (Y_1(n))_{11}, \tag{3.5}$$

where the matrices $Y_1(n)$ and $Y_2(n)$ are the first and second coefficients of the asymptotic series (3.3) and $Y_s(n)_{kj}$ denotes the k, j entry of the matrix $Y_s(n)$. Equations (3.4) and (3.5) reduce the problem of determining the asymptotic of the recurrence coefficients when $n = N$, $N\to\infty$ to the problem of the asymptotic solution of the RHP (1)–(3). In the case of a fixed external field $V(\xi)$, this analysis is performed in [10]. The approach in [10] consists of a succession of steps which, in the end, yields a reduced RHP for a matrix $M(\xi)$ and the behaviour of the coefficients γ_N and β_{N-1} as $N \to \infty$ can be recovered from $M(\xi)$. The following results can be found in [10].

Theorem 3.1. *The coefficients γ_N and β_{N-1} behave as $N \to \infty$*

$$\gamma_N = \gamma_N^0 + O(1/N), \qquad \beta_{N-1} = \beta_{N-1}^0 + O(1/N),$$

where

$$\left(\gamma_N^0\right)^2 = (M_1)_{12}(M_1)_{21}, \tag{3.6}$$

$$\beta_{N-1}^0 = \frac{(M_2)_{21}}{(M_1)_{21}} - (M_1)_{11}, \tag{3.7}$$

and M_1 and M_2 are 2×2 matrices which are recovered from the unique solution $M(\xi)$ of the following 2×2 matrix RHP:

$$M_+(\xi) = M_-(\xi)\nu(\xi), \qquad \xi \in \mathbb{R}, \tag{3.8}$$

$$M(\xi) = I + \sum_{k=1}^{\infty} \frac{M_k}{\xi^k}, \qquad \xi \to \infty, \tag{3.9}$$

and the matrix $\nu(\xi)$ is defined as

$$\nu(\xi) = \begin{pmatrix} \mathrm{e}^{-\mathrm{i}N\Omega_j} & 0 \\ 0 & \mathrm{e}^{\mathrm{i}N\Omega_j} \end{pmatrix}, \qquad \xi \in (u_{2j}, u_{2j+1}), \quad j = 1, \ldots, g, \tag{3.10}$$

$$\nu(\xi) = I, \qquad \xi \in (-\infty, u_1) \cup (u_{2g+2}, \infty), \tag{3.11}$$

$$\nu(\xi) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad \xi \in \bigcup_{j=1}^{g} (u_{2k-1} - u_{2k}). \tag{3.12}$$

We remark that theorem 3.1 holds true in the regular case, but not in singular cases where the equilibrium measure vanishes at the interior points of the spectrum or to higher order at the end points of the spectrum.

The solution of the RHP (3.8)–(3.9) derived in [10] can be rewritten in terms of the Szegö kernel of the surface X in the following way [13, 21].

On a Riemann surface $\mathcal{C}$, the Szegö kernel $S\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right](Q, P)$ is defined for all non-singular characteristics $\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right]$ as the $\left(\frac{1}{2}, \frac{1}{2}\right)$-form on $\mathcal{C} \times \mathcal{C}$ which has only a pole on the diagonal [17], namely as $P \to Q$

$$S\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(Q, P) = \frac{\sqrt{\mathrm{d}x(P)}\sqrt{\mathrm{d}x(Q)}}{x(P) - x(Q)}\left[1 + O((x(P) - x(Q))\right], \tag{3.13}$$

where x is a local coordinate. The Szegö kernel transforms when the variable P goes around a_k- and b_k-cycles as follows:

$$S\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(Q, P + a_k) = \epsilon^{2\pi \mathrm{i}\delta_k} S\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(Q, p), \tag{3.14}$$

$$S\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(Q, P + b_k) = \epsilon^{-2\pi \mathrm{i}\epsilon_k} S\begin{bmatrix}\delta\\ \epsilon\end{bmatrix}(Q, P), \qquad k = 1, \ldots, g. \tag{3.15}$$

The Szegö kernel of the hyperelliptic curve X can be written in the form [17]

$$S\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(P_0,P)=\frac{1}{2}\left(\frac{\gamma(\xi(P))}{\gamma(\xi(P_0))}+\frac{\gamma(\xi(P_0))}{\gamma(\xi(P))}\right)\frac{\theta\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right]\left(\int_{P_0}^{P}\boldsymbol{\omega};\Pi\right)}{\theta\left(\int_{P_0}^{P}\boldsymbol{\omega}\right)}\frac{\theta(\mathbf{0})}{\theta\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right](\mathbf{0})}\frac{\sqrt{\mathrm{d}\xi(P)\,\mathrm{d}\xi(P_0)}}{\xi(P)-\xi(P_0)},\tag{3.16}$$

where by $\xi(P)$ we still denote the projection map $P=(\xi,y)\to\xi$ from X to $\mathbb{C}_\infty$,

$$\gamma(\xi)=\sqrt[4]{\frac{\prod_{k=1}^{g+1}(\xi-u_{2k})}{\prod_{k=1}^{g+1}(xi-u_{2k-1})}},$$

and $\theta\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right](\boldsymbol{z})$ is the θ-function with the characteristics defined via

$$\theta\left[\begin{smallmatrix}\delta\\ \epsilon\end{smallmatrix}\right](\boldsymbol{z})=\sum_{\boldsymbol{n}\in\mathbb{Z}^g}\exp(\pi\mathrm{i}\langle B\boldsymbol{n}+B\boldsymbol{\delta},\boldsymbol{n}+\boldsymbol{\delta}\rangle+2\pi\mathrm{i}\langle\boldsymbol{z}+\beta,\boldsymbol{n}+\boldsymbol{\delta}\rangle).\tag{3.17}$$

For $P=(\xi,y)$, $P_0=(\xi_0,y_0)$, we define the quantity $\hat{S}(P,P_0)$ as

$$\hat{S}\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(P_0,P)=S\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(P_0,P)\frac{\xi-\xi_0}{\sqrt{\mathrm{d}\xi\,\mathrm{d}\xi_0}}.$$

Then the solution of the matrix RHP (3.8)–(3.9) takes the form

$$M(\xi)=\begin{pmatrix}\hat{S}\left[\begin{smallmatrix}\mathbf{0}\\ \frac{N}{2\pi}\Omega\end{smallmatrix}\right](\infty^1,P^1,) & \hat{S}\left[\begin{smallmatrix}\mathbf{0}\\ \frac{N}{2\pi}\Omega\end{smallmatrix}\right](\infty^1,P^2)\\ \hat{S}\left[\begin{smallmatrix}\mathbf{0}\\ \frac{N}{2\pi}\Omega\end{smallmatrix}\right](\infty^2,P^1) & \hat{S}\left[\begin{smallmatrix}\mathbf{0}\\ \frac{N}{2\pi}\Omega\end{smallmatrix}\right](\infty^2,P^2)\end{pmatrix},\tag{3.18}$$

where $P^{1,2}=(\xi,\pm y)$ are conjugate points on the Riemann surface X and $\infty^{1,2}=(\infty,\pm\infty)$. We remark that the path of integration between the points on different sheets of the Riemann surface X like

$$\int_{\infty^1}^{P^2}\boldsymbol{\omega}$$

is taken from ∞^1 to u_{2g+2} on the first sheet and from u_{2g+2} to P^2 on the second sheet. The entries of the matrix M do not have poles. Indeed, let us consider M_{11},

$$\hat{S}\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(\infty^1,P^1)=\frac{1}{2}\left(\gamma(\xi(P^1))+\frac{1}{\gamma(\xi(P^1))}\right)\frac{\theta\left(\int_{\infty^1}^{P^1}\boldsymbol{\omega}+\frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta\left(\int_{\infty^1}^{P^1}\boldsymbol{\omega}\right)}\frac{\theta(\mathbf{0})}{\theta\left(\frac{N}{2\pi}\boldsymbol{\Omega}\right)}.\tag{3.19}$$

The properties of the Szegö kernel guarantee that the g-zeros of $\theta\left(\int_{\infty^1}^{P^1}\boldsymbol{\omega}\right)$ in the denominator of the above expression are cancelled by the g-zeros of the term $\gamma(\xi(P^1))+\frac{1}{\gamma(\xi(P^1))}$, so the whole expression in (3.19) does not have poles but only singularities at u_k's of the type $1/\sqrt[4]{\xi-u_k}$. The same considerations can be done for the other entries of the matrix M. To verify (3.9), we observe that

$$\gamma(\xi(P^1))=\sqrt{\frac{\prod_{k=1}^{g+1}(\xi-u_{2k-1})}{y}},\qquad P^1=(\xi,y),$$

so that $\gamma(\xi(P^2))=-\mathrm{i}\gamma(\xi(P^1))$ and $\gamma(\infty^1)=1$. It then follows that

$$\hat{S}\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(\infty^1,\infty^1)=\hat{S}\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(\infty^2,\infty^2)=1,\qquad\text{or}\qquad M_{11}(\infty)=M_{22}(\infty)=1$$

and

$$\hat{S}\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(\infty^2,\infty^1)=\hat{S}\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(\infty^1,\infty^2)=0,\qquad\text{or}\qquad M_{12}(\infty)=M_{21}(\infty)=0.$$

The regular expansion of M in powers of $1/\xi$ as $\xi \to \infty$ follows from the fact that the point at infinity is a regular point of the Riemann surface X. To show that (3.8) is satisfied, let us denote by

$$\int_Q^P \boldsymbol{\omega}_\pm, \qquad \xi(P), \xi(Q) \in \mathbb{R},$$

the integrals on $\mathbb{C}_\pm$, namely the upper and lower part of the complex plane with respect to the real axis.

Then the following relations hold for $P^{1,2} = (\xi, \pm\sqrt{R(\xi)})$:

$$\left(\int_{\infty^{1,2}}^{P^1} \boldsymbol{\omega}_+ - \int_{\infty^{1,2}}^{P^2} \boldsymbol{\omega}_-\right)\bigg|_{\xi\in(u_{2k-1},u_{2k})} = -\sum_{j=k}^{g} \int_{a_j} \boldsymbol{\omega}, \quad k = 1, \ldots, g, \tag{3.20}$$

$$\left(\int_{\infty^{1,2}}^{P^1} \boldsymbol{\omega}_+ - \int_{\infty^{1,2}}^{P^2} \boldsymbol{\omega}_-\right)\bigg|_{\xi\in(u_{2g+1},u_{2g+2})} = 0, \tag{3.21}$$

$$\left(\int_{\infty^{1,2}}^{P^1} \boldsymbol{\omega}_+ - \int_{\infty^{1,2}}^{P^1} \boldsymbol{\omega}_-\right)\bigg|_{\xi\in(u_{2k},u_{2k+1})} = \int_{\beta_k} \boldsymbol{\omega}, \quad k = 1, \ldots, g, \tag{3.22}$$

$$\left(\int_{\infty^{1,2}}^{P^1} \boldsymbol{\omega}_+ - \int_{\infty^{1,2}}^{P^1} \boldsymbol{\omega}_-\right)\bigg|_{\xi\in(-\infty,u_1)} = 0, \tag{3.23}$$

where $\infty^{1,2}$ stands for ∞^1 or ∞^2. Similar obvious relations hold when the end point of the integration in (3.22) and (3.23) is P^2. Regarding the behaviour of the function γ, we have that

$$\gamma(\xi(P^1))_+ = \mathrm{i}\gamma(\xi(P^1))_-, \qquad \xi(P^1) \in J. \tag{3.24}$$

Combining (3.20)–(3.24) and the periodicity properties (3.14)–(3.15) of the Szegö kernel, it is straightforward to verify that expression (3.18) satisfies condition (3.8).

The entries of the matrix M_1, that is, the first term of the expansion of $M(\xi)$ as $\xi \to \infty$ are

$$(M_1)_{sr} = \hat{S}\begin{bmatrix} \mathbf{0} \\ \frac{N}{2\pi}\boldsymbol{\Omega} \end{bmatrix}(\infty^r, \infty^s) = \frac{\mathrm{i}(-1)^s}{4}\sum_{k=1}^{g+1}(u_{2k} - u_{2k-1})\frac{\theta\left[\begin{smallmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{smallmatrix}\right]\left(\int_{\infty^s}^{\infty^r}\boldsymbol{\omega}\right)}{\theta\left(\int_{\infty^s}^{\infty^r}\boldsymbol{\omega}\right)}\frac{\theta(\mathbf{0})}{\theta\left[\begin{smallmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{smallmatrix}\right](\mathbf{0})}, \tag{3.25}$$

for $s = 1, r = 2$ or $r = 1, s = 2$ and

$$(M_1)_{11} = \sum_{k=1}^{g} \partial_{z_k} \log\theta\left(\frac{N}{2\pi}\boldsymbol{\Omega}\right)\omega_k(\infty^1). \tag{3.26}$$

The entry 21 of the matrix M_2, that is, the second term of the expansion of $M(\xi)$ as $\xi \to \infty$ takes the form

$$(M_2)_{21} = \frac{\mathrm{i}}{4}\sum_{k=1}^{g+1}(u_{2k} - u_{2k-1})\frac{\theta(\mathbf{0})}{\theta\left(\frac{N}{2\pi}\boldsymbol{\Omega}\right)}\sum_{k=1}^{g}\frac{\partial}{\partial z_k}\left(\frac{\theta\left(\boldsymbol{v} + \frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta\left(\boldsymbol{v}\right)}\right)\omega_k(\infty^1), \tag{3.27}$$

where

$$\boldsymbol{v} := \int_{\infty^2}^{\infty^1} \boldsymbol{\omega}.$$

From (3.6)–(3.7) and (3.25)–(3.27), expressions (1.14) and (1.15) for γ_N^0 and β_N^0 can be obtained in a straightforward way, respectively.

In order to verify relations (1.16)–(1.18) the following Fay's identity [17] which relates the Szegö kernel and the canonical symmetric 2-differential is fundamental:

$$-S\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(P,Q)S\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(Q,P)=\mathcal{B}(P,Q)+\sum_{k=1}^{g}\frac{\partial^2}{\partial z_i\partial z_j}\log\theta\begin{bmatrix}\boldsymbol{\delta}\\ \boldsymbol{\epsilon}\end{bmatrix}(\mathbf{0})\omega_i(P)\omega_j(Q). \quad (3.28)$$

Proposition 3.2. *The coefficient γ_N^0 defined in (1.14) satisfies the relation*

$$\frac{1}{N^2}\frac{\partial^2}{\partial t_1^2}\log\left[\mathrm{e}^{-N^2F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right]=\left(\gamma_N^0\right)^2+O(1/N). \quad (3.29)$$

Proof. To prove the proposition, it is sufficient to multiply Fay's identity (3.28) by ξ and η and take the residue at $P=\infty^1$ and $Q=\infty^2$. The lhs gives

$$-\operatorname*{Res}_{P=\infty^1}\operatorname*{Res}_{Q=\infty^2}\left(\xi\eta S\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(P,Q)S\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(Q,P)\right)=-\left(\gamma_N^0\right)^2,$$
$$P=(\xi,y)\in X,\qquad Q=(\eta,w)\in X,$$

because of (1.14) and (3.25) and the rhs gives

$$\operatorname*{Res}_{P=\infty^1}\operatorname*{Res}_{Q=\infty^2}\left(\xi\eta\left[\mathcal{B}(P,Q)+\sum_{n,m=1}^{g}\frac{\partial^2}{\partial z_n\partial z_m}\log\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\omega_m(P)\omega_n(Q)\right]\right)+O\left(\frac{1}{N}\right)$$
$$=-\frac{1}{N^2}\frac{\partial^2}{\partial t_1^2}\log\left[\mathrm{e}^{-N^2F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right],$$

because of (2.27). □

To prove relation (1.17), we rewrite (1.11) in the form

$$\frac{1}{N^2}\frac{\partial^2\ln Z_N}{\partial t_1\partial t_2}=\gamma_N^2(\beta_{N-1}+\beta_N)=\gamma_N^2\left(2\beta_{N-1}-\frac{1}{N}\frac{\partial\log\gamma_N^2}{\partial t_1}\right),$$

where we have used the following relation in the last identity:

$$\frac{1}{N}\frac{\partial\log\gamma_n^2}{\partial t_1}=(\beta_{n-1}-\beta_n), \quad (3.30)$$

which follows from (1.8). Despite the formulae for γ_N^0 and $\beta_{N_1}^0$ being proved only for the fixed external field $V(\xi)$, we assume that they hold true while varying $V(\xi)$ in a sufficiently small range.

Proposition 3.3. *The following relation is satisfied:*

$$\frac{1}{N^2}\frac{\partial^2}{\partial t_1\partial t_2}\log\left[\mathrm{e}^{-N^2F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right]=\left(\gamma_N^0\right)^2\left(\beta_{N-1}^0+\beta_N^0\right)+O(1/N). \quad (3.31)$$

Proof. Using expressions (1.14) and (1.15), we obtain

$$\left(\gamma_N^0\right)^2\left(2\beta_{N-1}^0-\frac{1}{N}\frac{\partial\log\left(\gamma_N^0\right)^2}{\partial v_1}\right)=\frac{1}{16}\left(\sum_{k=1}^{g+1}(u_{2k}-u_{2k-1})\right)^2\frac{\theta(\mathbf{0})^2}{\theta\left(\frac{N}{2\pi}\boldsymbol{\Omega}\right)^2}$$
$$\times\left(\frac{\theta\left(-\boldsymbol{v}+\frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta(-\boldsymbol{v})}\sum_{j=1}^{g}\frac{\partial}{\partial z_j}\frac{\theta\left(\boldsymbol{v}+\frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta(\boldsymbol{v})}\omega_j(\infty^1)\right.$$

$$
-\frac{\theta\left(\boldsymbol{v}+\frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta(\boldsymbol{v})}\sum_{j=1}^{g}\frac{\partial}{\partial z_j}\frac{\theta\left(-\boldsymbol{v}+\frac{N}{2\pi}\boldsymbol{\Omega}\right)}{\theta(-\boldsymbol{v})}\omega_j(\infty^1)\Bigg)+O(1/N)
$$

$$
=\operatorname*{Res}_{P=\infty^1}\operatorname*{Res}_{Q=\infty^2}\left(\xi\eta^2 S\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(P,Q)S\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(Q,P)\right),
$$

$$
P=(\xi,y)\in X,\qquad Q=(\eta,w)\in X. \tag{3.32}
$$

A comparison of (2.27), when $Q=\infty^1$ is replaced by $Q=\infty^2$, (3.28) and (3.32) gives the statement. □

Finally, we prove the last relation (1.18).

Proposition 3.4. *The following relation is satisfied:*

$$
\frac{1}{N^2}\frac{\partial^2}{\partial t_2^2}\log\left[\mathrm{e}^{-N^2F_0}\theta\left(\frac{N\boldsymbol{\Omega}}{2\pi}\right)\right]=\left(\gamma_N^0\right)^2\left(\left(\gamma_{N-1}^0\right)^2+\left(\gamma_{N+1}^0\right)^2\right.
$$

$$
\left.+\left(\beta_N^0\right)^2+2\beta_N^0\beta_{N-1}^0+\left(\beta_{N-1}^0\right)^2\right)+O(1/N). \tag{3.33}
$$

Proof. Using relations (3.30) and

$$
\frac{\partial}{\partial t_1}\beta_N=\gamma_N^2-\gamma_{N+1}^2, \tag{3.34}
$$

which can be recovered from (1.9), we rewrite the lhs of (1.12) in the form

$$
\frac{1}{N^2}\frac{\partial^2\ln Z_N}{\partial t_2^2}=\gamma_N^2\left[\frac{1}{N^2}\frac{\partial^2\log\gamma_N^2}{\partial t_1^2}+2\gamma_N^2+\left(2\beta_{N-1}-\frac{1}{N}\frac{\partial\log\gamma_N^2}{\partial v_1}\right)^2\right]. \tag{3.35}
$$

Then, inserting the leading terms γ_N^0 and β_{N-1}^0 defined in (1.14) and (1.15) into the above relation and dropping terms of order $O(1/N)$ or higher, we arrive at the expression

$$
\left(\gamma_N^0\right)^2\left[\frac{1}{N^2}\frac{\partial^2\log\left(\gamma_N^0\right)^2}{\partial t_1^2}+2\left(\gamma_N^0\right)^2+\left(2\beta_{N-1}^0-\frac{1}{N}\frac{\partial\log\left(\gamma_N^0\right)^2}{\partial v_1}\right)^2\right]+O(1/N)
$$

$$
=\operatorname*{Res}_{P=\infty^1}\operatorname*{Res}_{Q=\infty^2}\left(\xi^2\eta^2 S\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(P,Q)S\begin{bmatrix}\mathbf{0}\\ \frac{N}{2\pi}\boldsymbol{\Omega}\end{bmatrix}(Q,P)\right),
$$

$$
P=(\xi,y)\in X,\qquad Q=(\eta,w)\in X,
$$

which by (3.28) and (2.27) proves the statement. □

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